Interactions and scaling in a disordered two-dimensional metal

Sudip Chakravarty[1], Lan Yin[1], Elihu Abrahams[2]
[1] *Department of Physics and Astronomy, University of California Los Angeles Los Angeles, CA 90095-1547*
[2] *Serin Physics Laboratory, Rutgers University, Piscataway, NJ 08855-0849*
(February 7, 2008)

We show that a non-Fermi liquid state of interacting electrons in two dimensions is stable in the presence of disorder and is a perfect conductor, provided the interactions are sufficiently strong. Otherwise, the disorder leads to localization as in the case of non-interacting electrons. This conclusion is established by examining the replica field theory in the weak disorder limit, but in the presence of arbitrary electron-electron interaction. Thus, a disordered two-dimensional metal is a perfect metal, but not a Fermi liquid.

A number of recent experiments [1] on the two-dimensional electron system in semiconductor devices have revealed the existence of a low temperature metal-insulator transition as the electron density is varied. In the 1980's, similar experiments appeared to confirm the prediction that the two-dimensional disordered electron system would have no metallic states [2]. What distinguishes the new samples from those studied earlier is that the electron density is extremely low, so that the Coulomb interaction energy is larger than the Fermi energy.

The combined effects of disorder and electron-electron interaction were studied by renormalization group (RG) methods [3,4] and there were indications that one effect of the interaction might be to stabilize a metallic state in two dimensions. However, the nature of the metallic state was unknown, and, in any case, the conclusion was not very definite since the RG flows went to a strong coupling regime beyond the range of validity of the RG equations. Therefore, the possibility of a two-dimensional metallic state remained an open issue from the theoretical point of view.

A scaling analysis of the recent experiments has been made [5], and, based on it, it was argued that any disordered two-dimensional metal is a perfect metal but not likely to be a Fermi liquid. The reason is that if the interaction is turned off, the electrons will localize [6], and the localized state has no resemblance to a Fermi gas. This motivated us to consider a two-dimensional non-Fermi liquid state and to study the effects of impurities.

As a minimal specification of a non-Fermi liquid, the retarded single particle Green's function must not contain a quasiparticle pole when analytically continued to the lower half plane, but a branch point. This leads to a spectral function satisfying the homogeneity relation [7,8]

$$A(\Lambda^{y_1}k, \Lambda^{y_2}\omega) = \Lambda^{y_A} A(k,\omega), \tag{1}$$

in the asymptotically low energy limit, where y_1, y_2, and y_A are the exponents defining the universality class of the non-Fermi liquid. Only the set of exponents $y_1 = 1$, $y_2 = 1$, and $y_A = -1$ represents a Fermi liquid for which the branch points collapse into simple poles. Here the momentum is measured with respect to k_F and the frequency is measured with respect to the Fermi energy.

The above spectral function will be assumed to contain a kinematic form factor of zero scale dimension, which is $\theta(\omega^2 - v_F^2 k^2)$. The rationale is as follows: if dissipation is due to the decay of an electron coupled to particle-hole pairs, then, for $\omega > 0$, ω has to be greater than $v_F|k|$. Similarly, because in the ground state of a non-Fermi liquid, particles are present both above and below the Fermi sea, ω must be less than $-v_F|k|$ for negative frequencies. The presence of the θ-function leads to a density of states that vanishes at the Fermi energy. In Ref. [7], another choice was made for which the density of states remains finite at the Fermi surface.

A spectral function with non-trivial exponents is necessary, but it does not fully specify a non-Fermi liquid. For example, it does not contain spin-charge separation, which requires separate singularities for spin and charge excitations. Moreover, new exponents may have to be introduced for the scaling of composite operators. It is, however, our intention to see what can be learned from this minimal specification of a non-Fermi liquid.

For explicit calculations we use the simple model:

$$A(\omega,k) \propto \frac{1}{\omega_c^{\alpha}|\omega - v_F k|^{1-\alpha}}\theta(\omega^2 - v_F^2 k^2), \tag{2}$$

where ω_c is a microscopic high frequency scale proportional to the inverse of the noninteracting density of states ν. The exponent α has to be positive to satisfy the analyticity properties required of the single particle Green's function. It is also worth noting that the singular part of the spectral function alone will not satisfy any sum rules.

The grassmannian field theory for the localization problem was first set up by Efetov, Larkin and Khmel'nitskii [9] and extended by Finkel'stein [3] to the

interacting case. We follow Ref. [3]. The random impurity potential is spatially uncorrelated and has a white-noise distribution with zero mean so that $\langle V(r)V(r')\rangle = (1/2\pi\nu\tau)\delta^{(d)}(r-r')$, The disorder average is carried out on the replicated partition function Z_N, where N is the number of replicas. It leads to the action

$$S = \sum_\alpha [S_0^\alpha + S_{\rm e-e}^\alpha] + R_0 \int d^d r \left(\overline{\psi}\psi\right)^2, \qquad (3)$$

where $\left(\overline{\psi}\psi\right) = \sum_{n,\alpha} \overline{\psi}_n^\alpha \psi_n^\alpha$. and $R_0 = 1/4\pi\nu\tau$. The Grassmann variables $\{\overline{\psi}_n^\alpha(r), \psi_n^\alpha(r)\}$ carry Matsubara index n ($\omega_n = \pi(2n+1)T$); the spin and the replica indices are lumped together in α. Here S_0^α is the action of free fermions. It and the electron-electron interaction term $S_{\rm e-e}^\alpha$ is diagonal in replica indices. Replica mixing takes place only in the impurity-induced interaction term.

If the electron-electron interaction leads to a Fermi liquid, the impurity induced interaction is relevant by power counting. Let us rewrite it as

$$R_0 \sum_{\alpha,\beta} \int d^d r d\tau d\tau' \overline{\psi}^\alpha(r,\tau)\psi^\alpha(r,\tau)\overline{\psi}^\beta(r,\tau')\psi^\beta(r,\tau') \quad (4)$$

Under the scale transformations $k \to \Lambda k$ and $\omega \to \Lambda\omega$, the dimension of the fermion variables obtained from the free fermion action is [10] $[\psi^\alpha(k,\omega)] \sim \Lambda^{-(3/2)}$ so that $[\psi^\alpha(r,\tau)] \sim \Lambda^{(1/2)}$. Note that spatial dimensionality, d, does not enter into the scale dimensions, because the scaling is in a direction normal to the Fermi surface; the transverse $(d-1)$ directions act as internal degrees of freedom. Then, from Eq. (4), the impurity-induced interaction scales as $R(l) = R_0 e^l$, where $\Lambda^{-1} = e^l$ and is thus relevant. Loop corrections show that either the system flows to a diffusive metal fixed point (a possibility that exists for $d > 2$) or to the Anderson localized phase. The possibility of a pure (ballistic) Fermi liquid phase does not exist because disorder is always a relevant operator [11].

In contrast, let us suppose that $S_0^\alpha + S_{\rm e-e}^\alpha$ leads to a non-Fermi liquid state. Consider the simple model in Eq. 2. Then it follows that $[\psi^\alpha(k,\omega)] \sim \Lambda^{-(\frac{3-\alpha}{2})}$, and $[\psi^\alpha(r,\tau)] \sim \Lambda^{\frac{1+\alpha}{2}}$. Consequently, $R(l) = R_0 e^{(1-2\alpha)l}$. Now the impurity-induced interaction is *irrelevant* for $\alpha > 1/2$ and relevant for $\alpha < 1/2$. For $\alpha > 1/2$, the disordered system is described by a non-Fermi liquid fixed point [11]. For $\alpha < 1/2$, the system has a choice between a diffusive metal or a Anderson localized phase.

To explore this situation further, we rewrite the action in terms of slow variables by isolating the regions of small total momentum in the replica-mixed particle-hole channel [3]. We define a matrix B in the replica, spin and Matsubara indices by $R_0\left(\overline{\psi}\psi\right)^2 = -\frac{1}{2}{\rm Tr}B^2(r)$. The Fourier transform of B is

$$B_{nn'}^{\alpha\beta}(q) = \frac{1}{\sqrt{2\pi\nu\tau\Omega}} \sum_k \overline{\psi}_n^\alpha(k)\psi_{n'}^\beta(k+q), \qquad (5)$$

where Ω is the sample volume. Introducing a hermitian matrix field $Q(r)$ to decouple this term [3], we find, using a trick due to Schwinger [12], which is valid even when the action is not quadratic in Grassmann variables, that

$$\overline{Z}_N \sim \int \mathcal{D}\{Q\} e^S = \int \mathcal{D}\{Q\} e^{-\frac{1}{2}\int d^d r \left[{\rm Tr}Q^2 + 2i\int_0^1 d\lambda \langle {\rm Tr}QB\rangle_\lambda\right]}, \qquad (6)$$

where we have taken the replica limit $N \to 0$ in a prefactor, and

$$\langle {\rm Tr}QB\rangle_\lambda = i\frac{d}{d\lambda}\log\left[\int \mathcal{D}\{\overline{\psi},\psi\} e^{\sum_\alpha (S_0^\alpha + S_{\rm e-e}^\alpha) - i\lambda\int d^d r {\rm Tr}QB}\right]. \quad (7)$$

From Feynman diagrams, one can see that electron-electron interaction cannot dress the QB vertices when the replica-off-diagonal components of Q are involved. However, if Q is diagonal, the vertices can receive corrections from electron-electron interaction. We shall neglect the vertex corrections for the replica-diagonal background field $Q^{\alpha\alpha}$. This approximation breaks gauge invariance, but has little effect on the localization problem, which crucially depends on the off-diagonal Q's. With this approximation [13], the quantity $\langle {\rm Tr}QB\rangle_\lambda$ can be calculated from the matrix Green's function $G_{mn}^{\alpha\beta}(r,r';\lambda)$, which satisfies the Dyson equation

$$G(r,r';\lambda) = G(r-r';0) - i\lambda\sqrt{2R_0}\int d^d r_1 G(r-r_1;0)Q(r_1)G(r_1,r';\lambda). \qquad (8)$$

The Green's function $G(r-r', i\omega_m;0)\delta_{mn}\delta_{\alpha\beta}$ is the fully interacting one, but in the absence of the matrix field $Q(r)$. It is translationally invariant and diagonal in Matsubara and replica indices. As a result of our approximation, $G(r-r';0)$ fully determines $G(r,r';\lambda)$ as a functional of $Q(r)$.

Consider the saddle point solution for which Q is a constant diagonal matrix, that is, $Q(r) = \overline{Q}(i\omega_n)\delta_{mn}\delta_{\alpha\beta}$ [8]. Then, the self-consistent equation for $\overline{Q}$ is

$$\overline{Q}(i\omega_n) = \frac{i}{\Omega}\sum_k \frac{\sqrt{2R_0}G(k,i\omega_n;0)}{1 + i\overline{Q}(i\omega_n)\sqrt{2R_0}G(k,i\omega_n;0)}. \qquad (9)$$

In the limit $(\nu/\tau) \ll 1$, we can write, to leading order,

$$\overline{Q}(i\omega_n) \approx \frac{i\sqrt{2R_0}}{\Omega}\sum_k G(k,i\omega_n;0) \qquad (10)$$

Using the explicit form of the spectral function given earlier, it is possible to show that

$$\overline{Q}(i\omega_n) = A(\alpha)\sqrt{\frac{\pi\nu}{2\tau}}\frac{1}{\cos(\alpha\pi/2)}|\omega_n/\omega_c|^\alpha {\rm sgn}(\omega_n), \quad (11)$$

where $A(\alpha)$ is a function of order unity. As the next iteration, one may substitute this expression for $\overline{Q}$ in the exact saddle point equation to estimate the correction. For $\alpha > 1/2$, it is readily seen that the correction vanishes in the limit of zero frequency. For $\alpha < 1/2$, the correction tends to infinity so that Eq. (11) cannot be a correct solution.

For $\alpha < 1/2$ and not too close to it, the saddle point value of $\overline{Q}$ can be calculated for the model spectral function from Eq. (9) without the approximation of Eq. (10). For the frequency tending to zero, it is

$$\overline{Q}(i\omega_n) = B(\alpha)\left(\frac{\pi\nu}{2\tau}\right)^{\frac{1}{2(1-2\alpha)}} \mathrm{sgn}(\omega_n) \equiv q\mathrm{sgn}(\omega_n), \quad (12)$$

where $B(\alpha)$ is a function of order unity, and we have neglected the weak frequency dependence. The saddle point solution for $\alpha < 1/2$ requires careful treatment of the cutoff. In the limit $\alpha \to 1/2$, the presence of the θ-function cutoff in the non-Fermi liquid spectral function cannot be ignored, while for $\alpha \to 0$ it can be. If this crossover is treated correctly, one finds that as $\alpha \to 1/2$,

$$\overline{Q}(i\omega_n) = \sqrt{2\pi\tau|\omega_n|}\left[(\exp(\frac{\pi\tau}{\nu}) - 1)^2 - (\frac{2\tau}{\nu})^2\right]^{-\frac{1}{4}} \mathrm{sgn}(\omega_n). \quad (13)$$

This solution smoothly joins the solution given in Eq. (12) at $\alpha = 1/2$.

To summarize: In the non-disordered case, the density of states for the non-Fermi liquid vanishes at zero frequency for all α. In the presence of disorder, this quantity is considered as an order parameter, and the non-Fermi liquid behavior survives if $\alpha > 1/2$, that is, when the the order parameter vanishes. If, on the other hand, $\alpha < 1/2$, the symmetry is broken, and there is a non-zero density of states in the disordered situation.

Next, we investigate the fluctuation modes about the mean-field solution. We consider a small deviation $\delta Q = (Q(r) - \overline{Q})$. It is useful to shift the variable further [3] by shifting $\overline{Q}$ to $\overline{Q} + H$, where

$$H(i\omega_n) = i\sqrt{2\pi\nu\tau}\left[G^{-1}(k, i\omega_n; 0) - G^{-1}(k, 0; 0)\right] I, \quad (14)$$

where I is the unit matrix in replica indices. Then it is not difficult to see that

$$S = -\mathcal{F}(\overline{Q} + H) - \frac{1}{2}\int d^d r \left[\mathrm{Tr}(\delta\tilde{Q})^2 + 2\mathrm{Tr}H\delta\tilde{Q}\right] + \delta S, \quad (15)$$

where $\delta\tilde{Q} = Q - \overline{Q} - H$. The quadratic part of δS is given by

$$\delta S_2 = R_0 \int \frac{d^d k}{(2\pi)^d} \delta\tilde{Q}^{\alpha\beta}_{mn}(k)\delta\tilde{Q}^{\beta\alpha}_{nm}(-k)\tilde{F}_{mn}(k), \quad (16)$$

with

$$\tilde{F}_{mn}(k) = \int \frac{d^d p}{(2\pi)^d}\tilde{G}(p, i\omega_m)\tilde{G}(p-k, i\omega_n). \quad (17)$$

Here

$$\tilde{G}(k, i\omega_n) = \frac{1}{G^{-1}(k, 0; 0)I + i\overline{Q}(i\omega_n)\sqrt{2R_0}}. \quad (18)$$

Note that H plays the role of a symmetry breaking field, which in the noninteracting case collapses to $\sqrt{2\pi\nu\tau}\omega_n I$. The shift, however, has allowed us to recast the expansion in terms of $\tilde{G}$ which is invariant under arbitrary homogeneous unitary rotations of the saddle point.

The $\delta\tilde{Q}$ fluctuation modes will be classified as longitudinal when $mn > 0$ and transverse when $mn < 0$. In the Fermi liquid case, the latter are the usual diffusion modes. The longitudinal modes are massive as in the non-interacting model [9]. The zero momentum limit of the transverse part of $\tilde{F}_{mn}(0)$ for $\alpha < 1/2$ is given by

$$\tilde{F}^T_{mn}(0) = \frac{1}{\Omega}\sum_k \frac{G^2(k, 0; 0)}{1 + 2R_0 q^2 G^2(k, 0; 0)}, \quad (19)$$

where q is defined in Eq. (12). One can verify using the saddle point equation that $\tilde{F}^T_{mn}(0) = (1/2R_0)$. This leads to a complete cancellation between the quadratic term $-\frac{1}{2}\int d^d r \mathrm{Tr}(\delta\tilde{Q})^2$ and δS_2 in the fluctuation action, Eq. (16). The resulting theory for the transverse modes is massless, and the remaining quadratic term in the action in the long wavelength limit is

$$\delta S_2 = -\frac{\pi\nu D(\alpha)}{4}\int \frac{d^d k}{(2\pi)^d} k^2 \delta\tilde{Q}^{\alpha\beta}_{mn}(k)\delta\tilde{Q}^{\beta\alpha}_{nm}(-k), \quad (20)$$

where the bare diffusion constant at the scale of the mean free path is renormalized by the electron-electron interaction and is ($\alpha < 1/2$ and not too close)

$$D(\alpha) = (1-\alpha)\left[\frac{\pi}{B(\alpha)}\right]^{\frac{2\alpha}{1-\alpha}}\left[\frac{2\tau}{\pi\nu}\right]^{\frac{2\alpha}{1-2\alpha}} D, \quad (21)$$

where $D = v_F^2\tau/d$ is the usual diffusion constant. In this equation we have also normalized the saddle point solution such that $\overline{Q}^2 = I$. As in the non-interacting problem, the resulting theory can now be cast in the language of a matrix nonlinear σ-model. It is well-known that all states are localized for $d \leq 2$. The only difference is that the bare diffusion constant $D(\alpha)$ is enhanced over the non-interacting value D.

As in the corresponding non-interacting problem, the mass term cancels in the transverse channel and the diffusion propagator diverges as k^{-2} at zero frequency, for $\alpha < 1/2$. These are properties of the Goldstone modes in the transverse replica channel and follow as long as the continuous symmetry in the Q-field theory is broken, which is indeed the case for $\alpha < 1/2$; they do not depend

critically on the assumptions made in the longitudinal channel. Thus, the neglect of the vertex correction in the longitudinal channel does not spoil this picture.

For $\alpha > 1/2$, $\overline{Q}(i\omega_n)$ tends to zero as the frequency tends to zero, and the transverse part of the mass term does not cancel against the transverse part of δS_2. Our analysis shows that impurity interactions are irrelevant for $\alpha > 1/2$, and the theory corresponds to the non-Fermi liquid fixed point Hamiltonian, minimally specified by the scaling property of the one-particle spectral function. It is preferable then to return to the original fermion language. Such a system should be a perfect metal.

It is important to make a number of qualifying remarks.

(1) For $\alpha > 1/2$, any attempt to expand in powers of the exponent α will lead to numerous logarithmic divergences, and the theory cannot be cast in the language of a Fermi liquid theory. For $\alpha < 1/2$, the electrons are localized in $d \leq 2$, with the interacting bare diffusion constant enhanced over the non-interacting value. Thus, once again, the language of Fermi liquid theory is not meaningful.

(2)The model spectral function considered here does not contain spin charge separation. From a simple model reflecting spin charge separation [7], it is possible to see that our qualitative conclusions regarding broken symmetry remain unchanged. It is not apparent, however, that that is the correct model for spin-charge separation. In fact, the entropy producing process in which an accelerated electron breaks up into a spin and a charge excitation is missing [14]. It is also possible that spin-charge separation will affect the behavior in a magnetic field. For example, in a two-dimensional system, a parallel magnetic field can couple to the spinon excitations, producing a gap in the spectrum, thereby destroying the non-Fermi liquid state. This, in turn, will lead to a localized insulating state as seen in experiments [15].

(3) The results obtained here are only superficially similar to those of the one-dimensional problem [16]. In one-dimensional theories, the anomalous dimension of the density-density correlation function cannot be simply obtained from the anomalous dimension of the one-particle Green's function. As remarked earlier, the operator product expansion of composite operators contain further singularities. Such possibilities were purposely avoided to reduce the complexity of the theory. Nonetheless, we have demonstrated that the breaking of the single most important assumption of the Fermi liquid theory, namely the existence of the quasiparticles, leads to dramatically different behavior with respect to the localization properties of the interacting electron system. Moreover, it appears that there are experimentally relevant regimes in two dimensions, which show no hint of instabilities that could arise from $2k_F$ vertex corrections. Thus, the neglect of vertex corrections in two dimensions may have a well-defined range of validity.

It is a pleasure to acknowledge important discussions with A. Balatsky, D. Belitz, T. Kirkpatrick, A. Ludwig, S. Sachdev, R. Shankar, H. Schulz, G. Zimanyi, S. C. Zhang, and especially C. Nayak. A part of this work was carried out at the Aspen Center for Physics. This work was supported by grants from the National Science Foundation: DMR-9531575 (S.C. and L.Y.) and DMR 9632294 (E.A.).

[1] S. V. Kravchenko *et al.*, Phys. Rev. Lett. **77**, 4938 (1996) and references therein; D. Popović, A.B. Folwer and S. Washburn, Phys. Rev. Lett. **79**, 1543 (1997).

[2] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).

[3] A.M. Finkel'stein, Zh. Eksp. Teor.Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)].

[4] For a review, see D. Belitz and T.R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).

[5] V. Dobrosavljević, E. Abrahams, E. Miranda, and S. Chakravarty, Phys. Rev. Lett. **79**, 455 (1997).

[6] P. A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[7] L. Yin and S. Chakravarty, Int. J. Mod. Phys. **10**, 805 (1996).

[8] It is sufficient to consider a translationally invariant definition as the replicated field theory is indeed translationally invariant.

[9] K. Efetov, A.I. Larkin and D.E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)].

[10] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).

[11] We are grateful to C. Nayak for clarifying remarks on these points.

[12] See, for example, H. M. Fried, *Functional methods and models in quantum field theory* (The MIT Press, Cambridge, 1972).

[13] This approximation fails in one dimension where such vertex corrections are governed by composite operators which are actually singular at momentum transfer $2k_F$.

[14] P. W. Anderson, *The theory of superconductivity in high-T_c cuprates* (Princeton University Press, Princeton, 1997).

[15] D. Simonian, S.V. Kravchenko, M.P. Sarachik, and V.M. Pudalov, Phys. Rev. Lett. **79**, 2304 (1997); V.M. Pudalov, G. Brunthaler, A. Prinz, and G. Bauer, Pis'ma Zh. Eksp. Teor. Fiz. **65**, 887 (1997) [JETP Lett. **65**, 932 (1997)].

[16] L.P. Gor'kov and I.E. Dzyaloshinskii, ZhETF Pis. Red. **18**, 686 (1973) [JETP Lett. **18**, 401 (1973)]; T. Giamarchi and H.J. Schulz, Phys. Rev. B **37**, 325 (1988).